NEEP 602 -- Engineering Problem Solving II
Exercise 9

Parabolic Partial Differential Equations

We'll begin our study of partial differential equations by considering 1-D, time-dependent problems. Our two independent variables will include one spatial variable and time. A typical problem involves heat flow in one dimension. In these problems, we will seek solutions for the temperature field in some 1-D region with specified boundary conditions, and an initial condition. The governing equation for these problems is:

$$\frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \nabla^2 T$$

where *T* is the temperature, ρ is the density, c_p is the heat capacity, and *k* is the thermal conductivity of the solid. In Cartesian coordinates this equation becomes:

$$\frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}.$$

As a model problem, we'll consider a uniform bar of length L which is initially at some uniform temperature T_0. We will assume that both ends of the bar ($x = 0$ and $x = L$) are then held at some temperature T_1 throughout the problem.

To solve this problem numerically, we divide the bar into several regions of width *h* and rewrite the equation in terms of the temperatures at each of these grid points and at each time step. We will label each temperature using the notation T_i^j, where *i* denotes the mesh point and *j* denotes the time step. When we difference the time derivative, we can write:

$$\frac{\partial T}{\partial t} \approx \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

where Δt is the time step. This is called a forward-differencing scheme because it uses the current and next time steps. The simplest way to solve the equation is to write the spatial derivative term at the current time step, giving:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{h^2}$$

We can substitute these approximations into the partial differential equation, giving

$$\left(\frac{\rho c_p}{k}\right)\frac{T_i^{j+1}-T_i^j}{\Delta t}=\frac{T_{i+1}^j-2T_i^j+T_{i-1}^j}{h^2}$$

Since there is only one term in this equation represents a temperature at the next time step, we can solve for that term in terms of temperatures at the current time step. Hence,

$$T_i^{j+1}=T_i^j(1-2\alpha)+\alpha(T_{i+1}^j+T_{i-1}^j)$$

where

$$\alpha=\frac{k\Delta t}{\rho c_p h^2}$$

This can be used to solve for all the temperatures at time step *j+1*, knowing the temperatures at time step *j*. No iteration is required. Unfortunately, there is a drawback to this method; it can be unstable if the time step and mesh spacing are not properly chosen. Specifically, we must have

$$\alpha\le\frac{1}{2}$$

in order to have a stable solution.

There is another technique that is unconditionally stable. This is the implicit technique, which is similar to the explicit technique described above except that the spatial derivative is written at the end of the step. That is,

$$\left(\frac{\rho c_p}{k}\right)\frac{T_i^{j+1}-T_i^j}{\Delta t}=\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{h^2}$$

Now we have several terms that represent temperatures at time step *j+1*. Solving for T_i^{j+1}, we obtain:

$$T_i^{j+1}=\frac{T_i^j+\alpha(T_{i+1}^{j+1}+T_{i-1}^{j+1})}{1+2\alpha}$$

This is easily implemented in a spreadsheet, but it does require iteration.