NEEP 602 -- Engineering Problem Solving II
Exercise 8

Eigenvalue Problems

Here we consider eigenvalue problems for ordinary differential equations. Applications of these types of problems include criticality calculations for fission reactor cores, solutions to partial differential equations solved using separation of variables, and vibration problems for structures. A typical problem would be:

$$\frac{d^2 y}{dx^2} + \lambda\ y = 0$$

with the boundary conditions

$$y(0) = y(1) = 0$$

One obvious solution to this problem is the trivial solution

$$y(x) = 0$$

but for certain values of λ, there are nontrivial solutions to this problem. These are called eigenvalues. The standard form of an eigenvalue problem is

$$\mathbf{AY} = \lambda \mathbf{Y}$$

or

$$[\mathbf{A} - \lambda \mathbf{I}]\mathbf{Y} = \mathbf{0}$$

where **Y** is an unknown column vector – the eigenvector corresponding to the eigenvalue λ, an unknown scalar.

To solve for the eigenvalues of our ODE analytically, we first determine the general solution, which is

$$y = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)\,.$$

Applying the first boundary condition ($y(0) = 0$) yields $B = 0$. Applying the second boundary condition gives

$$0 = A\sin(\sqrt{\lambda})$$

This gives us a solution for the eigenvalues:

$$\lambda_n = n^2 \pi^2 ,$$

where n is any integer. The solution functions for each eigenvalue are called eigenfunctions. For our model problem, the eigenfunctions are:

$$y_n = A_n \sin(n\pi x) .$$

Note that there are an infinite number of eigenvalues and that for each eigenvalue there is a distinct eigenfunction. Now we seek ways to solve these problems numerically.

Finite Differences

One solution method involves using finite differences. We can difference the differential equation in the usual way to obtain

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + \lambda\, y_i = 0 ,$$

which can be written as

$$y_{i-1} - (2 - \lambda\, h^2) y_i + y_{i+1} = 0$$

or, multiplying by -1 so that we can put it in the standard form (and have our eigenvalues typically come out positive),

$$-y_{i-1} + (2 - \lambda\, h^2) y_i - y_{i+1} = 0$$

If we write this equation at each of our mesh points and use the fact that $y = 0$ at each endpoint, then we can write this as a matrix equation corresponding to

$$\mathbf{[A - \lambda I]Y = 0}$$

as follows:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} [y] - \lambda h^2 [y] = 0 .$$

Now we can use Matlab's function `eig(matrixname)` to determine the eigenvalues of the matrix resulting from the differencing of our ODE, and then determine our eigenvalues by solving for them from

$$\lambda_i = e_i / h^2 ,$$

where e_i is the *i-th* eigenvalue of the matrix. Even though we set up the difference equations just as we did when solving for a particular solution y in our previous (non-eignevalue) problems, the eigenvalues of this matrix are those corresponding to all of the solutions (eigenvectors, or eigenfunctions) of our ODE.

Note that this would be the equation for a system which is divided up into 8 divisions, yielding a total of 9 mesh points, with the 7 internal mesh points being ones for which the dependent variable is unknown. Thus we have a 7x7 matrix and the vector [y] consists of 7 elements, each representing the unknown values of y at the internal mesh points.This gives us only a finite number of eigenvalues. If we desire more, we must use a smaller mesh spacing. Also note that the greater the number of mesh points that we take, the more accurate will be our results for the eigenvalues.