NEEP 602 -- Engineering Problem Solving II
Exercise 7

# Solving Boundary Value Problems in which the Boundary Conditions Involve a First Derivative

Our model problem for this discussion will be:

$$\frac{d^2y}{dx^2} + y = 0$$

with the two boundary conditions:

$$y(0) = 1$$
$$\frac{dy}{dx}(1) = 0 \, .$$

The solution to this differential equation is $y(x) = \tan(1)\sin(x) + \cos(x)$ .

The process is much the same as before. We approximate the second derivative with a finite difference formula and then substitute into the equation. This leaves us with:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = 0$$

To handle the derivative boundary condition at x=1, we must also write a finite difference approximation for the first derivative there. If $y_N$ is the value of y at the last mesh point, then we can write

$$\frac{dy}{dx}\Big|_{x=1} \approx \frac{y_{N+1} - y_{N-1}}{2h}$$

Since the slope should be 0 at x=1, then we can write

$$y_{N+1} - y_{N-1} = 0$$

This is then our boundary condition. This is a little strange, since $y_{N+1}$ is beyond the end of our region of interest, but it still let's us get a 0-slope boundary condition. The approach for implementing this depends on the tool we're using.

## Handling the Boundary Conditions in Excel

To handle the boundary condition at x=1 using Excel, we have at least two possible approaches. The first writes the regular finite difference formula for the differential equation at x=1 and then substitutes the boundary condition to remove the reference to the mesh point which is beyond the boundary. The process is something like this:

$$\frac{y_{N+1} - 2y_N + y_{N-1}}{h^2} + y_N = 0$$
$$y_{N-1} - (2 - h^2)y_N + y_{N+1} = 0$$
$$2y_{N-1} - (2 - h^2)y_N = 0$$
$$y_N = \frac{2y_{N-1}}{2 - h^2}$$

So instead of putting a value in the last cell, we put this formula there and we should end up with a slope of 0.

The other approach just puts the regular finite difference formula in the last cell and then adds an additional cell outside this boundary. This cell is then equated to the cell just inside the true boundary cell and then the iteration is begun.

## Handling the Boundary Conditions in MatLab

To solve the same problem using MatLab, the easiest approach is to just add one extra equation to our system and solve it. If we use 5 divisions to solve the following problem:

$$\frac{d^2y}{dx^2} + y = 0$$

with the two boundary conditions:

$$y(0) = 1$$
$$y(1) = 0.$$

then we can do it by solving the following matrix equation:

$$\begin{bmatrix} -(2-h^2) & 1 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 \\ 0 & 1 & -(2-h^2) & 1 \\ 0 & 0 & 1 & -(2-h^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is easily solved in Matlab. To solve the same differential equations with the new boundary conditions:

$$y(0) = 1$$
$$\frac{dy}{dx}(x = 1) = 0$$

we just introduce a new equation and now solve for 5 unknowns instead of 4. The fifth unknown is the solution for y at the right boundary. The system now becomes:

$$\begin{bmatrix} -(2-h^2) & 1 & 0 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 & 0 \\ 0 & 1 & -(2-h^2) & 1 & 0 \\ 0 & 0 & 1 & -(2-h^2) & 1 \\ 0 & 0 & 0 & 2 & -(2-h^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$