NEEP 602 -- Engineering Problem Solving II
Exercise 4
Spring 2005

Stiff ODEs

The concentration of a reactant $y(t)$ in a combustion process can be modeled with the ODE

$$y' = y^2(1-y) \quad y(0) = \varepsilon$$

To see what a stiff ODE solution looks like, solve this IVP over an interval $0 \le t \le 2/\varepsilon$. Use a value of 1e-4 for ε, and solve it with the Matlab solver ode15s (the ‘s’ stands for stiff). If you don’t specify the output arguments in the solver call, Matlab will automatically plot the solution as it computes it. Use a statement like this

```
ode15s(@func,timespan,initial);
```

As you’ll see, the solution is rather fast, and the value of the concentration increases rapidly to a value approaching 1. It stays near its limit of 1 for the remainder of the interval.

Now, save a copy of your script for this problem. In this copy, use the solver ode45, which is not designed for stiff problems. This solver can generate the solution, but it really has to work at it! In order to get some idea of how hard ode45 is working compared to ode15s, add the following to your script:

```
options = odeset(‘Stats’, ‘on’);
```

and include `options` in the input arguments for your ode15s and ode45 calls. This will show the number of successful steps, failed steps, and function evaluations, plus several other statistics from each solution.

You might try decreasing the relative error tolerance in the ode45 script, to see how asking for more accuracy can really slow things down sometimes. You can also try the other, less accurate Matlab stiff solver ode23s.

ODEs Involving a Mass Matrix

Sometimes it is convenient to write ODEs in the form

$$M(t,y)y' = F(t,y)$$

where $M(t,y)$ is a nonsingular mass matrix that allows us to collapse the equations of motion into one equation. Matlab can handle ODEs in this form with the use of the option `Mass`. The following equations of motion are used to model the motion of two pallets slung beneath a Chinook helicopter as a double pendulum (Giordano & Weir (1991)).

$$(m_1 + m_2)L_1\theta_1'' + m_2L_2\cos(\theta_2 - \theta_1)\,\theta_2'' - m_2L_2(\theta_2')^2\sin(\theta_2 - \theta1) + (m_1 + m_2)\,g\sin\theta_1 = 0$$

$$m_2L_2\theta_2'' + m_2L_1\cos(\theta_2 - \theta_1)\,\theta_1'' + m_2L_1(\theta_1')^2\sin(\theta_2 - \theta1) + m_2g\sin\theta_2 = 0$$

Here θ_1 is the angular position of the upper pallet, and θ_2 is the angular position of the lower pallet. The equation in matrix form is shown in the attached sheet.

The equations can be linearized as follows:

$$(m_1+m_2)L_1\theta_1'' + m_2L_2\,\theta_2'' + (m_1+m_2)\,g\,\theta_1 = 0$$

$$m_2L_2\theta_2'' + m_2L_1\,\theta_1'' + m_2g\,\theta_2 = 0$$

The matrix equation is also attached. The analytical solution of the linearized equations can be shown to be

$$\theta_1(t) = -\tfrac{1}{2}\cos(2t) - \tfrac{1}{2}\sin(2t)$$

$$\theta_2(t) = \cos(2t) + \sin(2t)$$

To make the problem more interesting, assume that the hook holding the lower pallet cable is open, so that if the lower pallet swings through an arc of more than 60°, the cable will slip off and the pallet will be lost. You can use event location to detect this occurence and terminate the solution.

Use the values for the parameters given below and solve both the linear and nonlinear equations over a time interval from $t = 0$ to $t = 2\pi$ and report whether or not the lower pallets falls off, and if so, at what time it does. Compare the numerical solution obtained from the linear model with that from the nonlinear model. Compare the solution from the linear model to the analytical solution.

Use $m_1 = 937.5$ slugs, $m_2 = 312.5$ slugs, $L_1 = L_2 = 16$ ft, and $g = 32$ ft/s^2 for the values of the parameters.

For initial conditions use

$$\theta_1(0) = -0.5,\ \theta'_1(0) = -1,\ \theta_2(0) = 1, \text{ and } \theta'_2(0) = 2$$

Double Pendulum – Linear

$$My' = f(t,y)$$

$$\mathrm{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1+m_2)L_1 & 0 & m_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & m_2L_1 & 0 & m_2L_2 \end{bmatrix}$$

$$\mathrm{M}y' = \mathrm{M}\begin{bmatrix} \theta_1' \\ \theta_1'' \\ \theta_2' \\ \theta_2'' \end{bmatrix} = \begin{bmatrix} \theta_1' \\ -(m_1+m_2)g\theta_1 \\ \theta_2' \\ -m_2g\theta_2 \end{bmatrix}$$

Double Pendulum – Nonlinear

$$\mathrm{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (m_1 + m_2)L_1 & 0 & m_2 L_2 \cos(\theta_2 - \theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & m_2 L_1 \cos(\theta_2 - \theta_1) & 0 & m_2 L_2 \end{bmatrix}$$

$$\mathrm{M}\mathrm{y}' \;=\; \mathrm{M}\begin{bmatrix} \theta_1' \\ \theta_1'' \\ \theta_2' \\ \theta_2'' \end{bmatrix} = \begin{bmatrix} \theta_1' \\ m_2 L_2 (\theta_2')^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1 \\ \theta_2' \\ -m_2 L_1 (\theta_1')^2 \sin(\theta_2 - \theta_1) - m_2 g \sin\theta_2 \end{bmatrix}$$