Ex. 11
4/19/04

NEEP 602 -- Engineering Problem Solving II
Exercise 11

Elliptic Partial Differential Equations (Plate Problems)

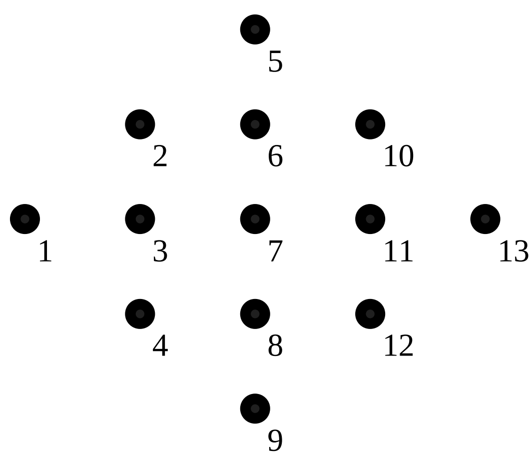
Here we consider a a particular type of elliptic PDE: plate theory. The deformation of plates is governed by the following fourth-order PDE:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

where *w* is the displacement of the plate, *q* is the transverse pressure applied to the plate, and

$$D = \frac{Et^3}{12(1-\upsilon^2)}$$

Here, *E* is the elastic modulus for the plate material, υ is its Poisson's ratio, and t is the plate thickness. To difference this equation, it's convenient to number the mesh points in the neighborhood of a particular point. I'll use the following numbering scheme:



Our goal is a finite difference approximation to our PDE, written about point 7. We can write the first term as:

$$\frac{d^4w}{dx^4} \approx \frac{\frac{d^2w}{dx^2}\Big|_3 - 2\frac{d^2w}{dx^2}\Big|_7 + \frac{d^2w}{dx^2}\Big|_{11}}{h^2}$$

$$= \frac{(w_1 - 2w_3 + w_7) - 2(w_3 - 2w_7 + w_{11}) + (w_7 - 2w_{11} + w_{13})}{h^4}$$

$$= \frac{w_1 - 4w_3 + 6w_7 - 4w_{11} + w_{13}}{h^4}$$

Similarly,

$$\frac{d^4w}{dx^2dy^2} \approx \frac{\frac{d^2w}{dx^2}\Big|_6 - 2\frac{d^2w}{dx^2}\Big|_7 + \frac{d^2w}{dx^2}\Big|_8}{h^2}$$

$$= \frac{\left(w_2 - 2w_6 + w_{10}\right) - 2\left(w_3 - 2w_7 + w_{11}\right) + \left(w_4 - 2w_8 + w_{12}\right)}{h^4}$$

Substituting these into our PDE yields the following difference equation for the operator:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\Big|_7 \approx$$

$$\frac{(w_1 + w_5 + w_9 + w_{13}) + 2(w_2 + w_4 + w_{10} + w_{12}) - 8(w_3 + w_6 + w_8 + w_{11}) + 20w_7}{h^4}$$

This can then be used to solve plate problems.