NEEP 602 -- Engineering Problem Solving II
Exercise 10

Elliptic Partial Differential Equations (2-D Boundary Value Problems)

Introduction

Elliptic partial differential equations can be thought of as two-dimensional boundary value problems. The classic problem is heat conduction in a plate, where boundary conditions are provided on the 4 sides of a plate and we seek the temperature distribution in the interior of the plate. Other examples of elliptic PDE's include diffusion, electric fields, and torsion. We will deal, initially, with heat conduction problems in two dimensions. In these problems, we will seek solutions for the temperature field in some 2-D region with specified boundary conditions. The governing equation for these problems is:

$$\nabla^2 T + \frac{Q}{k} = 0$$

where *T* is the temperature, *Q* is the volumetric heating, and *k* is the thermal conductivity of the solid. In Cartesian coordinates this equation becomes:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{Q}{k} = 0$$

while in polar coordinates the governing equation is:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{Q}{k} = 0$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{Q}{k} = 0\,.$$

As a model problem, we'll consider a rectangular plate of width *a* and height *b*, so our region of interest is the rectangle defined by: $0<x<a$ and $0<y<b$. For boundary conditions, we will assume we know the temperature along the entire boundary of this plate. These are known as Dirichlet boundary conditions. We will assume that the temperature on all the boundaries is zero, except for the boundary at $y=0$, on which we

will assume the temperature to be *T(x,0)=f(x)*, where *f(x)* is a known function. This equation can be solved using separation of variables, giving the solution:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)$$

$$a_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)$$

$$T(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left[\frac{n\pi(b-y)}{a}\right]\text{cosech}\left(\frac{n\pi b}{a}\right)$$

To find the solution, we determine the coefficients a_n and then sum the series for the temperature field. This is actually done more easily with numerical methods.

To solve this problem numerically, we divide the rectangle into a grid and assign a temperature to each grid point. This is the 2-D equivalent of dividing a 1-D region into intervals to create mesh points. We will label each temperature using the notation $T_{i,j}$, where *i* denotes the row number and *j* denotes the column number. We will assume that the grid spacing in the *x*-direction is given by *h* and the spacing in the *y*-direction is *s*. Given this model, we can approximate the second partial derivative of the temperature with respect to *x* as:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2}$$

and the second partial derivative of the temperature with respect to *y* as:

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{s^2}$$

If we substitute these into our PDE and set *Q=0*, then we obtain:

$$\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{h^2} + \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{s^2} = 0$$

Solving this for $T_{i,j}$ we obtain:

$$T_{i,j} = \frac{T_{i,j+1} + T_{i,j-1} + \frac{h^2}{s^2}\left(T_{i+1,j} + T_{i-1,j}\right)}{2\left(1 + \frac{h^2}{s^2}\right)}.$$

For the case of equal mesh spacing in each direction, *i.e.* $h=s$, we obtain:

$$T_{i,j} = \frac{T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j}}{4}.$$

In other words, the temperature at any mesh point is the average of its four nearest neighbors.

Now that we have done 1-D boundary value problems, you can easily see how we might use Excel to solve this type of problem. We just set up a grid of cells representing the temperatures at each of the grid points, put appropriate formulas in these cells relating any cell to its neighbors, set the boundary temperatures, and then have the spreadsheet iterate until a solution is reached.