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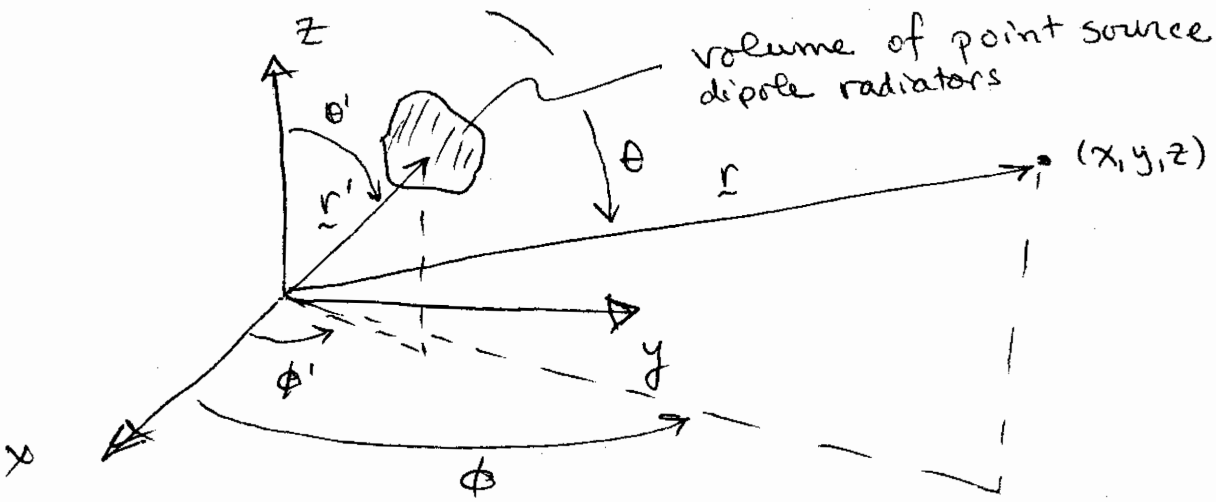
3-D Wave Green's Function

Example:

- scalar Helmholtz Eq.
- 3 D
- open (unbounded)
- spherical (r, θ, ϕ)



for example, fill the volume with dipoles, all polarized in $\hat{z}$:

$$\nabla^2 A_z + \beta_0^2 A_z(r,\theta,\phi) = -\mu_0 J_z(r,\theta,\phi) \qquad \text{SCALAR H-H eq.}$$

$$\nabla^2 G + \beta_0^2 G = \delta(\underline{r} - \underline{r}')$$

This one really benefits from experience...

Brute force approach will be something like Section 14.6.3, although that case specifically looks at a bounded spherical wave problem. Nevertheless, very ugly math.

Alternative
Clever approach:

(1) Derive G for a point source at the origin first,

$$\delta(\underline{r}-\underline{r}') \rightarrow \delta(\underline{r}) \qquad (\underline{r}'=0)$$

then,

(2) Generalize to case with point source at $\underline{r}' \neq 0$.

$$\nabla^2 G(\underline{r},\underline{r}') + \beta_0^2 G(\underline{r},\underline{r}') = \delta(\underline{r})$$

$G(\underline{r} \rightarrow \infty)$ = outward-traveling wave

$|G(\underline{r} \rightarrow \infty)| < \infty$

For a point source at $\underline{r}'=0$, there are no angular features to the source & since no boundaries, G, here, must be independent of ϕ, θ angles.

Hence,

$$\nabla^2 G + \beta_o^2 G = \frac{d^2 G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \beta_o^2 G = \frac{1}{4\pi r^2}\delta(r).$$

$$\text{or} \quad r^2 \frac{d^2 G}{dr^2} + 2r\frac{dG}{dr} + (\beta_o r)^2 G = \frac{\delta(r)}{4\pi}$$

Since $\underline{r}' = 0$, here, the "closed-form approach" only requires one basis solution (for $r > 0$) to the homogeneous 1D equation:

$$r^2 \frac{dG}{dr^2} + 2r\frac{dG}{dr} + (\beta_o r)^2 G = 0$$

$$G(r) = A\frac{e^{-j\beta r}}{r} + \cancel{B\frac{e^{+j\beta r}}{r}} = A\frac{e^{-j\beta r}}{r}$$

for outward-propagating wave if assuming $e^{j\omega t}$ time convention.

to evaluate A, one can integrate this equation over an infinitesimal volume that surrounds the point source at the origin & then allow $r \to 0$. Doing this yields

$$\boxed{A = -\frac{1}{4\pi}}$$

Hence

$$G(r)\Big|_{\substack{\text{point}\\ \text{source}\\ \text{at origin}}} = -\frac{e^{-j\beta_0 r}}{4\pi r} .$$

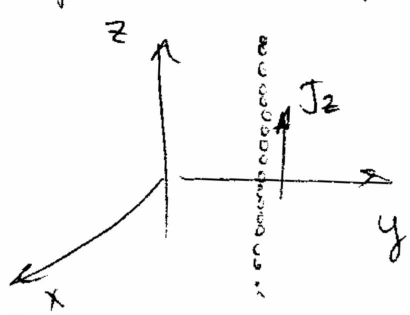
Now, removing this point source from the origin, but keeping the same physics intact, one simply obtains:

$$\boxed{G = -\frac{1}{4\pi}\,\frac{e^{-j\beta_0 |\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}}$$

Scalar H-H equation Green's Function for point source at $\underline{r}=\underline{r}'$!

(note: I would not expect all of you to have guessed at the trick, here)

question: what if we imagine constructing an infinite unity-strength line source (uniform along z) from a superposition of point sources:



... then a $\underline{\underline{2D}}$ Green's Function for this problem can be obtained as:

$$-\frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{e^{-j\beta_0 |\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}\, dz = G(\rho, \phi, \rho', \phi')$$

However, from Eq. (11-28a) in the textbook, we can learn that

$$\int_{-\infty}^{+\infty} \frac{e^{-j\beta_0 R}}{R}\, dz = -j\pi H_0^{(2)}(\beta_0 R)$$

$$R \equiv |\underline{r}-\underline{r}'|$$

thus,

$$\boxed{G(\rho, \phi, \rho', \phi') = +\frac{j}{4} H_0^{(2)}(\beta_0 R)}$$

which is just what we had already concluded on page 410 of the notes!