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2-Dimen. Wave Green's Function

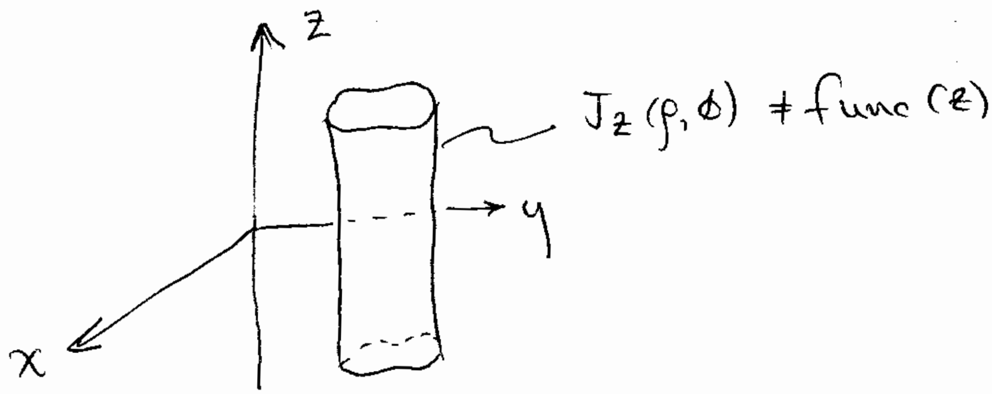
Example:

- Scalar Helmholtz Eq
- 2D
- open (unbounded)
- cylindrical (ρ, ϕ)

e.g., Example 14-5

$\Leftrightarrow$ solving problems of the form

$$\nabla^2 A_z(\rho, \phi) + \beta_0^2 A_z = -\mu_0 J_z(\rho, \phi)$$



relevant "BC's"

(1) $|A_z(\rho=0)| < \infty$

(2) $|A_z(\rho \to \infty)| \to 0$ (like ripples in a pond wherein a stone is dropped, the intensity falls off as the ripple radius gets large)

(3) $A_z(\phi=0) = A_z(\phi=2\pi)$

(4) solutions represent (radially) outward propagating waves

Seek a <u>closed</u>-form solution for the ρ (radial) dependence ... Fourier expansion for ϕ dependence ...

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2} + \beta_0^2 G = \frac{1}{\rho}\delta(\rho-\rho')\,\delta(\phi-\phi')$$

again, assume a separable form solution:

$$G(\rho,\phi,\rho',\phi') = \sum_m g_m(\rho)\,.\,\psi_m(\phi)$$

here, experience indicates choosing

$$\psi_m(\phi) = e^{jm\phi} \qquad m = -\infty,\ldots -1,0,1,\ldots \infty$$

is wise.

Then

$$G(\rho,\phi,\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho)\, e^{jm\phi}$$

Substitute this into, multiply both sides

by $e^{-jn\phi}$, integrate from $\int_0^{2\pi} d\phi\, e^{j(m-n)\phi} = \begin{cases} 2\pi, & m=n \\ 0, & m\neq n \end{cases}$

(orthogonality for $e^{jm\phi}$)

And obtain that each of the expansion "coefficient functions" $g_m(\rho)$ satisfies (407)

$$\rho \frac{d^2 g_m}{d\rho^2} + \frac{dg_m}{d\rho} + \left(\beta_0^2 \rho - \frac{m^2}{\rho}\right) g_m = \frac{e^{-jm\phi'}}{2\pi} \delta(\rho-\rho')$$

Now, obtain $g_m(\rho)$ using "closed-form" recipe.

first, identify this with Sturm-Liouville form

with
$$p(\rho) = (\rho)^{2\pi}/e^{-jm\phi'}$$
$$q(\rho) = (m^2/\rho) \times 2\pi/e^{-jm\phi'}$$
$$r(\rho) = \rho \times 2\pi/e^{-jm\phi'}$$
$$\lambda = \beta_0^2$$

And the homogeneous equation is Bessel's equation,

$$\rho \frac{d^2 g_m}{d\rho^2} + \frac{dg_m}{d\rho} + \left(\beta_0^2 \rho - \frac{m^2}{\rho}\right) g_m = 0$$

A simple solution for $\rho < \rho'$ that satisfies $|g_m^{(1)}(0)| < \infty$ is

$$\boxed{g_m^{(1)}(\rho) = J_m(\beta_0 \rho), \quad \rho < \rho' \text{ \& satisfies } \rho = 0 \text{ B.C.}}$$

A <u>simple</u> solution for $\rho > \rho'$ that satisfies $g_m^{(2)}(\rho \to \infty)$ = <u>outward</u> propagating wave is:

$$\boxed{g_m^{(2)} = H_m^{(2)}(\beta_o \rho) \quad , \quad \rho > \rho' \quad \text{\& satisfies B.C. at } \rho = \infty.}$$

Hankel Function of the 2ND kind (see App. IV)

Now, evaluate Wronskian $W(\rho)$ @ $\rho = \rho'$ ...

$$W(\rho = \rho') = \beta_o \left[J_m(\beta_o \rho') H_m^{(2)\prime}(\beta_o \rho') - H_m^{(2)}(\beta_o \rho') J_m'(\beta_o \rho') \right]$$

Now, using some addition theorems & other Bessel Function Identities, can be written as:

$$W(\rho') = -j\beta_o \left[J_m(\beta_o \rho') Y_m'(\beta_o \rho') - J_m'(\beta_o \rho') Y_m(\beta_o \rho') \right],$$

<u>But</u>: $J_n(\alpha x) Y_n'(\alpha x) - Y_n(\alpha x) J_n'(\alpha x) = \dfrac{2}{\pi \alpha x}$

So that

$$\boxed{W(\rho') = -j \frac{2}{\pi \rho'}}$$

Applying the recipe for the closed-form Gr. Func.,

$$g_m(\rho) = \begin{cases} J_m(\beta_o \rho) H_m^{(2)}(\beta_o \rho') e^{-jm\phi'} & , \quad 0 \le \rho \le \rho' \\ J_m(\beta_o \rho') H_m^{(2)}(\beta_o \rho) e^{-jm\phi'} & , \quad \rho' \le \rho \le \infty \end{cases}$$

Hence,

$$G(\rho,\phi,\rho',\phi') = -\frac{1}{4j}\begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} J_m(\beta_o\rho)\, H_m^{(2)}(\beta_o\rho')\, e^{jm(\phi-\phi')}, & \rho<\rho' \\ \displaystyle\sum_{m=-\infty}^{+\infty} J_m(\beta_o\rho')\, H_m^{(2)}(\beta_o\rho')\, e^{jm(\phi-\phi')}, & \rho>\rho' \end{cases}$$

However, it is possible to show that

$$\sum_{n=-\infty}^{+\infty} J_n(\beta\rho)\, H_n^{(2)}(\beta\rho')\, e^{jn(\phi-\phi')} = H_o^{(2)}[\beta|\rho-\rho'|]$$

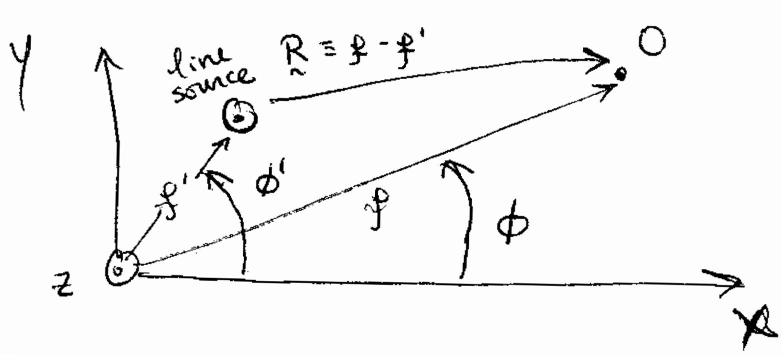
as well as

$$\sum_{n=-\infty}^{+\infty} J_n(\beta\rho')\, H_n^{(2)}(\beta\rho)\, e^{jn(\phi-\phi')} = H_o^{(2)}[\beta|\rho'-\rho|]$$

$$= H_o^{(2)}[\beta|\rho-\rho'|]$$

where

$$|\rho-\rho'| = \sqrt{\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')}$$



Hence,

$$\boxed{G(\rho,\phi;\rho',\phi') = -\frac{1}{4j} H_0^{(2)}(\beta_0|\underline{\rho}-\underline{\rho}'|) \text{ , for all } \rho\,!}$$

discontinuity at $\rho \gtrless \rho'$ handled by use of absolute value in argument of $H_0^{(2)}$ for the separation distance.

Well-known form of 2D, cylindrical wave problem Green's Function.

- very compact!

Recall (p.935, in textbook):

$$H_0^{(2)}(\beta_0|\underline{\rho}-\underline{\rho}'|)\Big|_{|\underline{\rho}-\underline{\rho}'|\to\infty} \simeq \sqrt{\frac{2}{\pi(\beta_0|\underline{\rho}-\underline{\rho}'|)}}\, e^{+j\frac{\pi}{4}}\, \underbrace{e^{-j(\beta_0|\underline{r}-\underline{r}'|)}}_{\text{outward-going wave}}$$

amplitude decreases with increasing $|\underline{\rho}-\underline{\rho}'|$ ✓

outward-going wave ✓