Review of Phasor Analysis

Sine waves are the basis for most generation, transmission and utilization of electric power. The ability to carry out computations for sinusoidally varying quantities is the foundation for much of power engineering analysis and these computations are universally based on the use of Phasors. While circuits courses which develop the phasor method are prerequisite to ECE 355, most students come into the course with a very rusty set of phasor skills. The following material is a very brief overview of the phasor concept along with a set of problems typical of what will be found in the course. I strongly urge all students to read over this material and, if necessary, revisit your circuits text if you have trouble with any of the material in these notes.

Basic Concept of Phasor Representation

Phasor analysis is based on Euler's equation which allows writing sine and/or cosine functions as exponential functions. The enormously simpler manipulation rules associated with exponential functions compared to trigonometric functions is what makes the method useful. Euler's Equation is

$$\varepsilon^{j\theta} = \cos\theta + j\sin\theta \qquad (1)$$

To represent the sinusoid

$$v(t) = V_m \cos(\omega t + \phi) \qquad (2)$$

we set $\theta = \omega t + \phi$ and use Euler's Equation to obtain

$$\varepsilon^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi) \qquad (3)$$

from which it is clear that the real part is what we need. Multiplying by V_m to account for the size of v(t) and taking the real part results in

$$v(t) = V_m \cos(\omega t + \phi) = \text{RE}\left[V_m \varepsilon^{j(\omega t + \phi)} \right] \qquad (4)$$

Using the law which says that the product of two exponentials is obtained by simply adding their exponents allows us to rewrite Eq 4 as

$$v(t) = V_m \cos(\omega t + \phi) = \text{RE}\left[V_m \varepsilon^{j\phi} \varepsilon^{j\omega t} \right] \qquad (5)$$

and this result expresses the one to one correspondence between the world of sinusoids at frequency ω and the world of complex numbers. On the left hand side of Eq 5 is the trigonometric function $V_m \cos(\omega t + \phi)$ characterized by the two parameters V_m and ϕ (for a particular, known frequency ω). On the right hand side these two parameters form the complex number $V_m \varepsilon^{j\phi}$ which completely characterizes the original sinusoid. We call this complex number $V_m \varepsilon^{j\phi}$ the *phasor* representation of the sinusoid. Because it utilizes the real part of the Euler' Equation it is often referred to as having a cosine reference (a pure cosine yields a phasor with only a real part since $\phi = 0$). It is also be possible to utilize the imaginary part and have a sine reference but this is less common. Also in power engineering the rms value of the current or voltage is of more significance than the peak value and the phasor is defined using the rms value as

$$\bar{\mathbf{V}} = V_{rms}\varepsilon^{j\phi} = V\varepsilon^{j\phi} \qquad \text{where } V = V_{rms} \qquad (6)$$

The time domain expression must therefore include a $\sqrt{2}$

$$v(t) = V_m \cos(\omega t + \phi) = \text{RE}\,[\sqrt{2}V_{rms}\,\varepsilon^{j\phi}\,\varepsilon^{j\omega t}] = \text{RE}\,[\sqrt{2}\bar{\mathbf{V}}\,\varepsilon^{j\omega t}] \qquad (7)$$

Graphical Interpretation

There is a simple graphical interpretation of Eq 5 which provides a means of keeping track of phase angles even when some of the sinusoidal quantities are expressed as sine waves and some as cosine waves. The phasor defined in Eq 6 is a complex number and can be graphically visualized as a line in the complex plane having a length equal to the rms value of the sinusoid and an angle equal to the phase angle of the sinusoid *when expressed as a cosine*. If the wave is a cosine wave, the phase angle is zero and the phasor lies along the real axis as shown in Fig 1. To recover the time function, Eq 7 indicates three steps:

1) multiply by $\sqrt{2}$ 2) multiply by $\varepsilon^{j\omega t}$ 3) take the real part.

The first is simply a scale factor to return to peak values. The second imparts rotation at a rate of w rad/sec to the phasor and creates a constant amplitude rotating line out of the stationary phasor. And the third says retain only the real part (projection of the rotating line) as a time function. This is illustrated in Fig 1 where the projection of the cos phasor clearly traces a cosine wave as the $\varepsilon^{j\omega t}$ term causes it to rotate. Following the same ideas, to create a sine wave the original phasor must lie on the *negative* j-axis as shown in the figure.

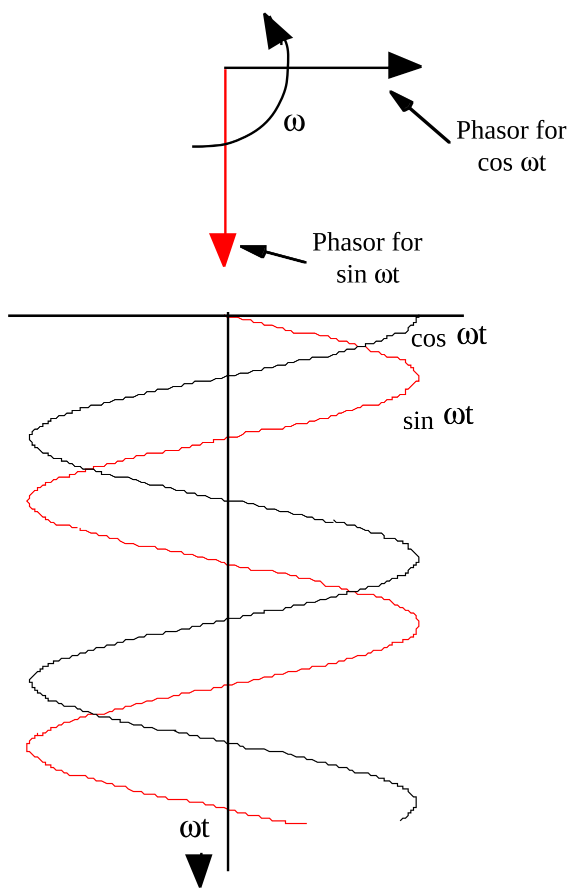
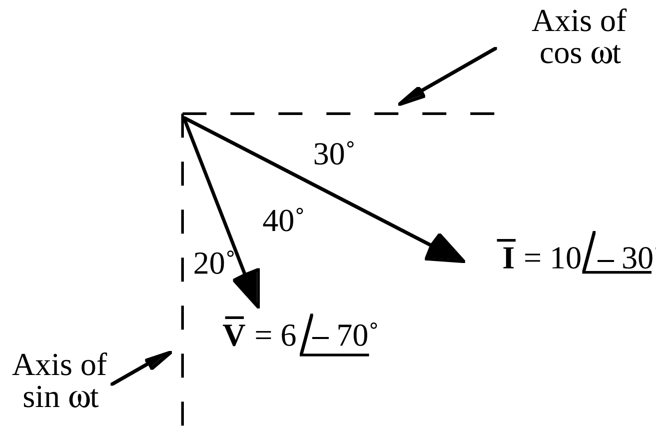


Figure 1 Graphical Relationship Between Phasor and Time Function

In addition to being a useful way of keeping the phasor-sinusoid relationship in mind, the phasors for cosine and sine allow easy conversion from any phasor to either a cosine or a sine referenced time function or the reverse, for any time function to a cosine or sine referenced phasor. Figure 2 illustrates this application.



$$i(t) = 14.14\cos(\omega t - 30^\circ) = 14.14\sin(\omega t + 60^\circ)$$

$$v(t) = 8.49\cos(\omega t - 70^\circ) = 8.49\sin(\omega t + 20^\circ)$$

Figure 2 Example of Use of Cos and Sin Axes to Relate Phasors and Sinusoids

Phasor Operations

Representation of sinusoids by complex numbers is nice but if it did not offer significant advantages in carrying out calculations and in visualizing results it would be only marginally useful. Because the basic operational rules are much simpler for exponentials than for sinusoids, computations using phasors are much, much simpler than for sinusoids. Consider, for example, the addition of two sinusoidal waves (which is a long, cumbersome procedure using trigonometry) by means of phasors. Writing out the sum in detail

$$v_1(t) + v_2(t) = \mathrm{RE}\left[\sqrt{2}\bar{\mathbf{V}}_\mathbf{1}\,\varepsilon^{j\omega t}\right] + \mathrm{RE}\left[\sqrt{2}\bar{\mathbf{V}}_\mathbf{2}\,\varepsilon^{j\omega t}\right] \qquad (8)$$

since when we add complex numbers we add their real parts and add their imaginary parts, it follows that

$$v_1(t) + v_2(t) = \mathrm{RE}\left[\sqrt{2}\bar{\mathbf{V}}_\mathbf{1}\,\varepsilon^{j\omega t} + \sqrt{2}\bar{\mathbf{V}}_\mathbf{2}\,\varepsilon^{j\omega t}\right] = \mathrm{RE}\left[(\sqrt{2}\bar{\mathbf{V}}_\mathbf{1} + \sqrt{2}\bar{\mathbf{V}}_\mathbf{2})\varepsilon^{j\omega t}\right] \qquad (9)$$

which shows that to find the phasor of the sum of two sinusoids we only need to add the phasors of the two sinusoids. The complicated trigonometric operations are replaced by a simple sum of complex numbers. Once this property is established we can use it whenever we need to perform a sum. Other such properties exist and a table of operations can be established relating sinusoids at a frequency ω and the corresponding phasors.

Table 1
Sinusoid - Phasor Relationships

Sinusoids		Phasors
$v(t) = \sqrt{2}V \cos(\omega t + \phi)$	$\leftrightarrow$	$\bar{\mathbf{V}} = V\,\varepsilon^{j\phi}$
$K\,v(t)$	$\leftrightarrow$	$K\,\bar{\mathbf{V}}$
$v_1(t) \pm v_2(t)$	$\leftrightarrow$	$\bar{\mathbf{V}}_1 \pm \bar{\mathbf{V}}_2$
$\frac{d}{dt}v(t)$	$\leftrightarrow$	$j\omega\,\bar{\mathbf{V}} = \omega V\,\varepsilon^{j(\phi + 90^\circ)}$
$\int v(t)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}\bar{\mathbf{V}} = \frac{1}{\omega}V\,\varepsilon^{j(\phi - 90^\circ)}$

No equivalence for multiplication of two sinusoids can be given - indeed multiplying two sine waves does not produce a sine wave at the same frequency but instead two new frequencies, the sum and difference, are produced. The operations in the table are sufficient for much of what is important in engineering systems employing sine waves. There is, however, one more significant phasor concept which is of great value in analysis.

Complex Impedance

Like the product, the ratio of two sinusoids is not a sinusoid of the same frequency and is not a useful quantity. However, the ratio of the phasors representing two sinusoids is again a complex number (not a phasor since it does not represent a sinusoid) and does have an important meaning in electrical engineering. In particular, the ratio of the phasor voltage to the phasor current at a pair of terminals is defined as the complex impedance.

$$\bar{\mathbf{Z}} = \frac{\bar{\mathbf{V}}}{\bar{\mathbf{I}}} = \frac{V\,\varepsilon^{j\phi}}{I\,\varepsilon^{j\alpha}} = Z\,\varepsilon^{j(\phi-\alpha)} \qquad (10)$$

Examination of Eq 10 shows that the amplitude of Z defines the ratio of V/I and the angle defines the phase difference between $\bar{\mathbf{V}}$ and $\bar{\mathbf{I}}$. The two quantities, Z and $\phi - \alpha$, characterize the volt-amp relation at the terminal pair in a concise manner. The complex impedance of the basic circuit elements is easily determined from the time domain volt-amp relations and the operations in Table 1.

$$v(t) = R\, i(t) \quad \leftrightarrow \quad \bar{\mathbf{V}} = R\, \bar{\mathbf{I}} \qquad \bar{\mathbf{Z}}_{\mathbf{R}} = R$$

$$v(t) = L \frac{d}{dt} i(t) \quad \leftrightarrow \quad \bar{\mathbf{V}} = j\omega L\, \bar{\mathbf{I}} \qquad \bar{\mathbf{Z}}_{\mathbf{L}} = j\omega L \qquad (11)$$

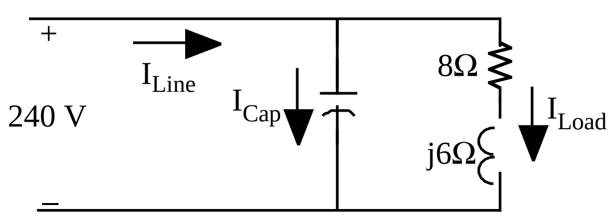
$$v(t) = \frac{1}{C} \int i(t)\, dt \quad \leftrightarrow \quad \bar{\mathbf{V}} = \frac{1}{j\omega C} \bar{\mathbf{I}} \qquad \bar{\mathbf{Z}}_{\mathbf{C}} = \frac{1}{j\omega C}$$

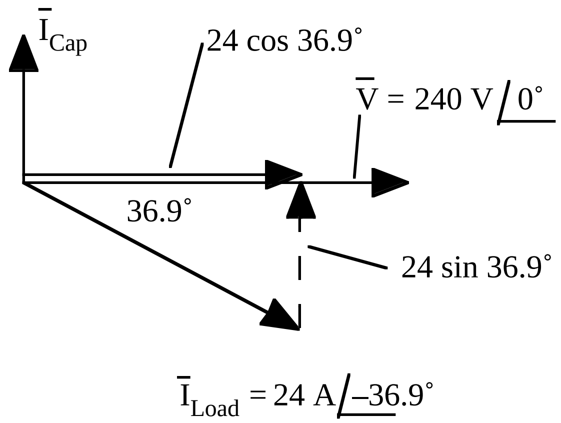
and all of the resistance based concepts of dc circuit theory (series/parallel combinations, Thevenin's Theorem, loop and node equations) apply to phasor analysis using the complex impedance.

Phasor diagrams

A phasor diagram is nothing more than a diagram illustrating how the various phasors in a particular situation relate to one another in satisfying the volt-amp relations of the circuit being studied. The diagram is simply a pictorial way of viewing the voltages and currents. The advantage is that one can often "see" how to approach the solution or reach a general conclusion about systems of a particular type. Two examples serve to illustrate the effective use of a phasor diagram.

Example 1 Power Factor Correction





A load with an impedance of 8 + j6 Ω is fed from a 240 volt, 60 hz source. It is desired to connect a capacitor in parallel as shown in the circuit above to cause the power factor to be unity (to have the source voltage and line current in phase.

Solution: The current taken by the load is

$$I_{Load} = \frac{240 \angle 0^\circ}{8 + j6} = 10 \angle -36.9^\circ$$

The voltage and load current are shown in the phasor diagram above. The capacitor current, 90˚ leading the voltage is also shown. In order to have the line current in phase with the voltage, it is clear from the diagram that the capacitor current magnitude must be equal to 24 sin 36.9˚ so it cancels the component of the load current perpendicular to the voltage and the resulting line current is 24 cos 36.9˚

$$I_{Cap} = 24 \sin 36.9^\circ = 14.4 \text{ A} \qquad I_{Line} = 24 \cos 36.9^\circ = 19.2 \text{ A}$$

The amplitude of the complex impedance and capacitance of the capacitor are therefore

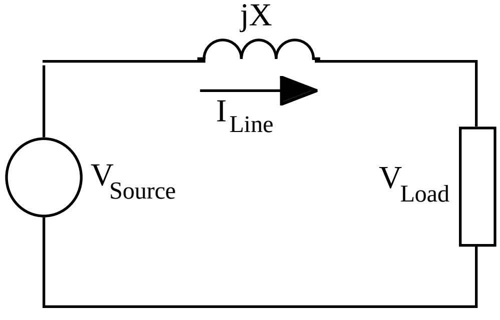
$$Z_{Cap} = \frac{240}{14.4} = 16.7 = \frac{1}{377\text{x}C} \qquad C = \frac{1}{377\text{x}16.7} = 158.8\ \mu\text{f}$$

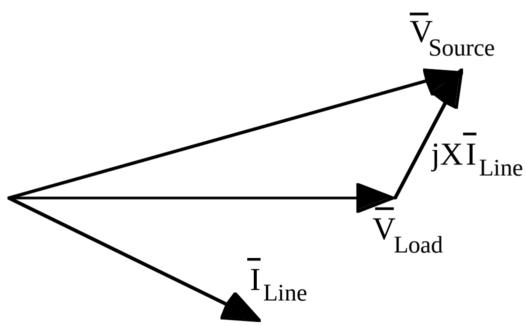
Example 2 Can a Load with Leading Current Cause Voltage Rise
It has been suggested that with a leading current in the load, the load voltage magnitude can be larger than the source voltage magnitude of a transmission line. Investigate this assertion and, if true, give a quantitative measure of how much lead angle is required to observe a voltage rise.

Solution: The transmission line impedance in a 60 hz system can often be approximated as purely reactive. The circuit of interest is that shown in the figure below. The phasor equation relating the source and load end voltages is

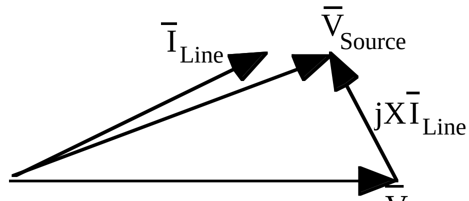
$$\bar{\mathbf{V}}_{\mathbf{Source}} = \bar{\mathbf{V}}_{\mathbf{Load}} + jX\, \bar{\mathbf{I}}_{\mathbf{Line}}$$

Using this equation the phasor diagrams for lagging line current and leading line current are also shown in the figure. It is clear from the diagrams that when the current is lagging the direction of the $jX\, \bar{\mathbf{I}}_{\mathbf{Line}}$ voltage phasor is always going to result in the source voltage magnitude exceeding the load voltage magnitude. When the current is leading, the $jX\, \bar{\mathbf{I}}_{\mathbf{Line}}$ voltage phasor rotates counter clockwise and if the lead angle is large enough can cause the load voltage magnitude to exceed the source voltage magnitude.



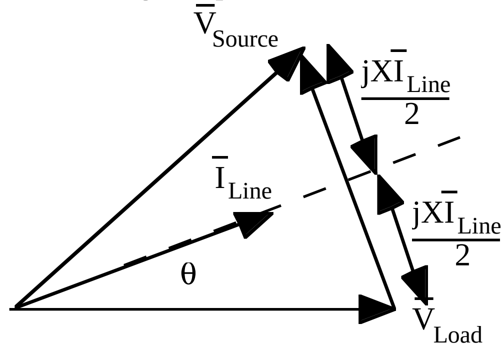


Lagging Current



Leading Current

The second part of the problem is to give a quantitative measure of how much lead angle is required to observe a voltage rise. This can be accomplished by sketching the phasor diagram for the boundary case where the source and load voltage amplitudes are the same.



$$V_{Source} = V_{Load} = V$$

In this case the voltage triangle is isosceles and the expression for the minimum required lead angle θ is

$$\sin\theta = \frac{\frac{XI_{Line}}{2}}{V} = \frac{XI_{Line}}{2V}$$

Phasor Review Problems

The Following Problems Illustrate Most of the Phasor Concepts Important in ECE 355

1) Express each of the following sinusoids as a phasor using the specified reference.

   a) 50 cos (ωt – 30˚) cosine reference

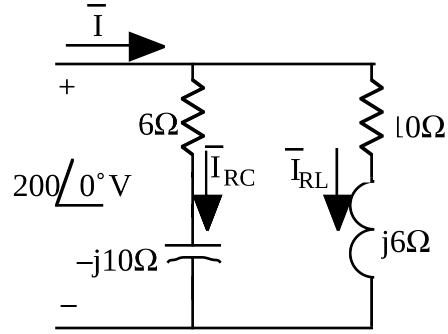
   b) 50 cos (ωt – 30˚) sine reference

   c) 100 sin (ωt + 30˚) cosine reference

   d) 100 sin (ωt + 30˚) sine reference

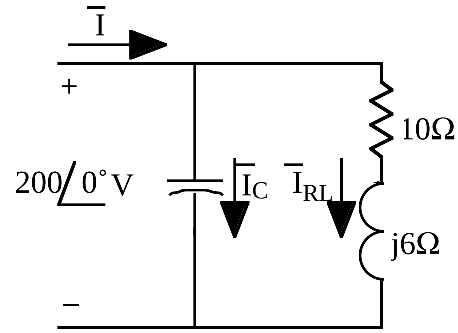
2) Find the sum of the sinusoids in a and c of problem 1 expressed as a phasor (using a cosine reference) and as a sinusoid.

3) Find the three currents in the figure below. Draw a phasor diagram showing the voltage as reference and the phasor addition which results in the current $\bar{I}$.

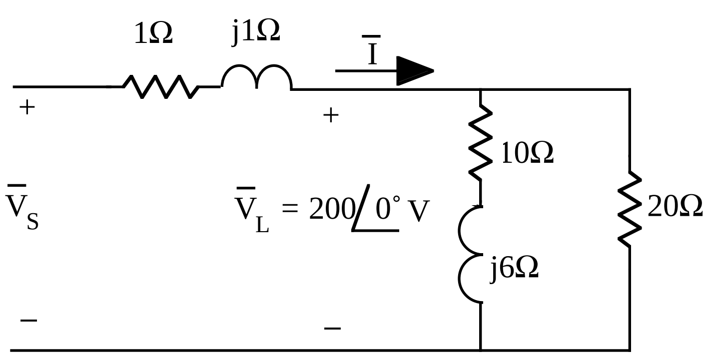


4) Find the average power and the reactive power associated with $\bar{I}_{RL}$, $\bar{I}_{RC}$ and $\bar{I}$ in problem 3.

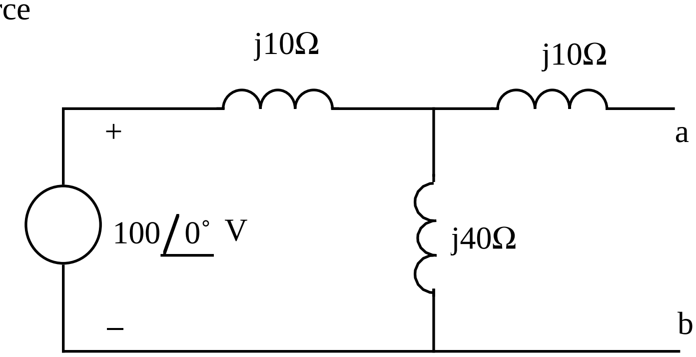
5) Find the value of the capacitor which will correct the power factor to 0.95 lagging in the circuit below. Assume the source frequency is 60 hz. Draw a phasor diagram showing the voltage as reference and the phasor addition which results in the improved power factor.



6) Find the line current $\bar{I}$ and the source voltage $\bar{V}_S$ in the circuit below. Draw a phasor diagram showing the load voltage $\bar{V}_L$ as reference and including the current $\bar{I}$, the voltage drop in the line impedance 1 + j1 and the source voltage $\bar{V}_S$.



7) a) Find the Thevenin equivalent circuit at the terminals a-b for the circuit below
b) Repeat a if the source is a 100 A current source



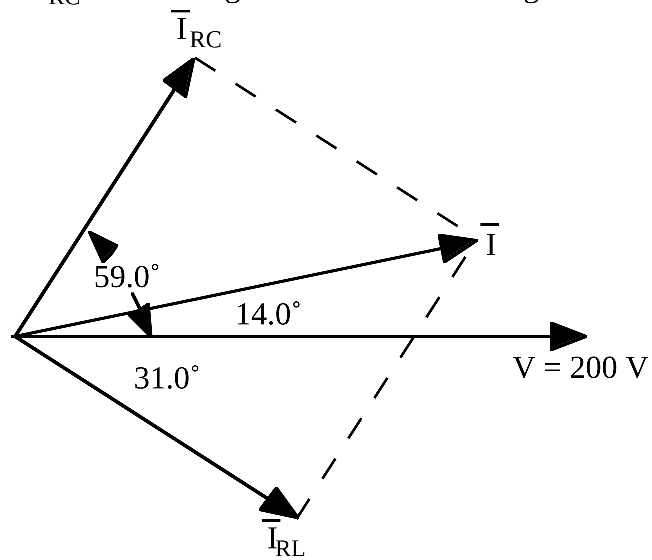
Phasor Review Problems

Answers

1) a) 35.4 angle –30˚
   b) 35.4 angle 60˚
   c) 70.7 angle –60˚
   d) 70.7 angle 30˚

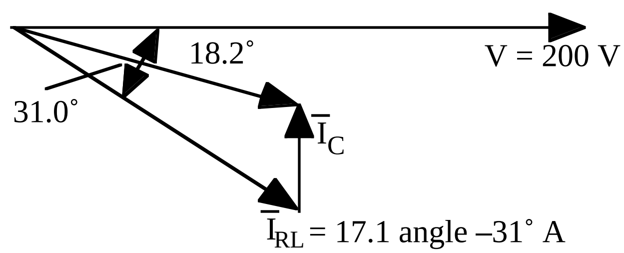
2) phasor = 66.0 – j78.9 = 102.8 angle –50.1
   sinusoid = 145.4 cos (ωt – 50.1)

3) $\bar{I}_{RL}$ = 17.15 angle –31.0˚ $\bar{I}_{RC}$ = 17.15 angle 59.0˚ $\bar{I}$ = 24.2 angle 14.0˚

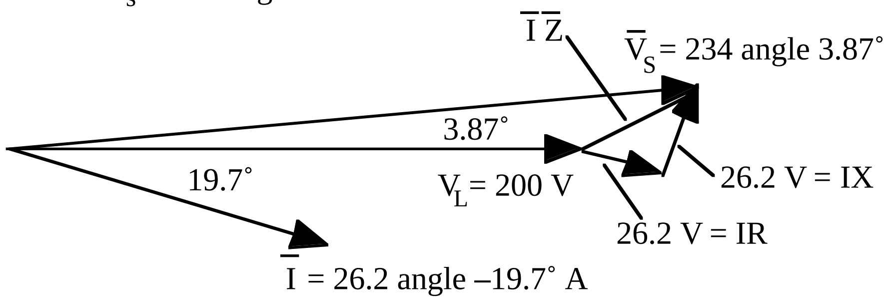


4) P_{RL} = 2940 W Q_{RL} = 1760 Var
   P_{RC} = 1760 W Q_{RL} = 2940 Var
   P_T = 4700 W Q_T = 1180 Var

5) C = 52.8 μf



6) $\bar{I}$ = 26.2 angle –19.7˚ $\bar{V}_S$ = 234 angle 3.87˚



7) a) $\bar{V}_{th} = 80 \angle 0^\circ$ $\bar{Z}_{th} = j18$

b) $\bar{V}_{th} = 4000 \angle 90^\circ$ $\bar{Z}_{th} = j50$