ECE334 **Spring 2006**

State Space Systems Analysis

Mid-term Exam

Thursday, March 9, 2006 — 11:00am-12:15pm

Name :	Solution

This is a closed book exam. Please show all working. Backs of pages may be used for scratch work if necessary. All answers should include units wherever appropriate.

Enjoy!

Prob 1	Prob 2	Prob 3	Prob 4	Prob 5	Prob 6	Prob 7	Total Score (70)

Problem 1 (10 points)

In the standard basis, a vector has the representation $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. In an alternative basis, its representation is $\tilde{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$. Determine a basis (any basis) that would give the representation $\tilde{x}$.

Let B define the alternative basis, with the columns of B being the basis's vectors. Then

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = B \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Note that the basis vectors must be linearly independent, which is true in this case.

Problem 2 (10 points)

The dynamic behaviour of a nonlinear system is described by

$$\dot{x}_1 = x_1^2 x_2 - 1$$
$$\dot{x}_2 = x_1 - x_2$$

This system has an equilibrium point at (1,1). Determine the linear model $\dot{x} = Ax$ that approximately describes dynamic behaviour in the neighbourhood of this equilibrium point.

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1x_2 & x_1^2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

At the point (1,1) this becomes,

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Problem 3 (10 points)

Determine the matrix that has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, and corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A = V \Lambda V^{-1} \quad \text{where} \quad V = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$V^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

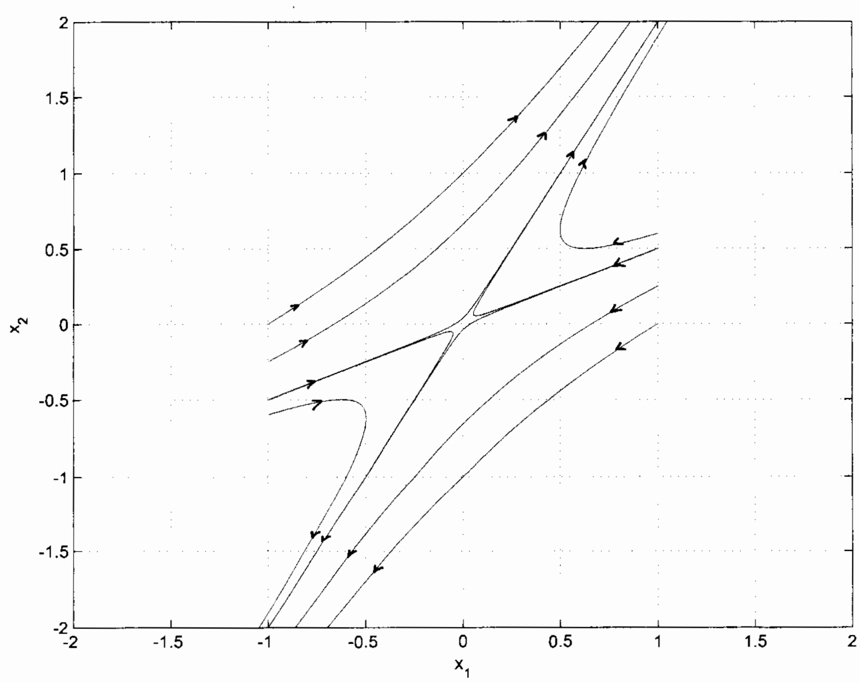
$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \left(-\frac{1}{3}\right)\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{5}{3} \end{bmatrix}$$

Problem 4 (10 points)

The matrix in Problem 3 is the A-matrix of a linear system $\dot{x} = Ax$. Sketch the phase portrait behaviour for this linear system.



Problem 5 (10 points)

Determine the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} -1.4 & -0.4 \\ -0.6 & -1.6 \end{bmatrix}$$

The eigenvalues are given by

$$|sI - A| = 0 \Rightarrow \begin{vmatrix} s+1.4 & 0.4 \\ 0.6 & s+1.6 \end{vmatrix} = 0$$

$$\Rightarrow (s+1.4)(s+1.6) - 0.24 = 0$$

$$\Rightarrow s^2 + 3s + 1.4 \times 1.6 - 0.24 = 0$$

$$\Rightarrow s^2 + 3s + 2 = 0$$

$$\Rightarrow (s+1)(s+2) = 0$$

Therefore the eigenvalues are $s_1 = -1$, $s_2 = -2$.

The eigenvectors are given by $(sI - A)u = 0$

For $s_1 = -1$,

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = 1, \; \alpha_2 = -1$$

$$\Rightarrow u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $s = -2$

$$\begin{bmatrix} -0.6 & 0.4 \\ 0.6 & -0.4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = 2, \; \alpha_2 = 3$$

$$\Rightarrow u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Problem 6 (10 points)

Does the vector $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ lie in the space spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$? Justify your answer.

If $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ lies in the space spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then they are linearly dependent, and we can find $\alpha_1, \alpha_2, \alpha_3 \neq 0$ such that

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives,

$$\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = -\alpha_2$$

$$\alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_2 = -2\alpha_3 \Rightarrow \alpha_1 = 2\alpha_3$$

$$2\alpha_1 + 2\alpha_3 = 0$$

$$\Rightarrow 4\alpha_3 + 2\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_2 = 0$$

Because $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the three vectors are linearly independent, so $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ does not lie in the space spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Problem 7 (10 points)

A system is described by the differential equation

$$2\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + y(t) = 4\frac{du}{dt} + 3u(t)$$

Rewrite the system description in state-space form.

The de in standard form is

$$\frac{d^3y}{dt^3} + \frac{3}{2}\frac{d^2y}{dt^2} + 0\frac{dy}{dt} + \frac{1}{2}y(t) = 2\frac{du}{dt} + \frac{3}{2}u(t)$$

Controllable canonical realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \frac{3}{2} & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable canonical realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ \frac{3}{2} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$