Problem Set 3

Prob 1 The desired subspace consists of $x \in \mathbb{R}^4$ such that

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The desired subspace is therefore the null space of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

The dimension of x is 4, and the rank of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is 2, therefore the desired subspace has dimension $4-2=2$.

$\Rightarrow$ the basis consists of 2 linearly independent vectors.

Using the Matlab command "null(A, 'r')" we obtain the basis that consists of the vectors

$$x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Prob 2 a) In the standard basis,

$$A \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 1 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -5 & -2 \\ 2 & 7 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

b) Let $\tilde{x}, \tilde{y}$ be the representations of x, y in the new basis.

Then,

$$x = B\tilde{x}, \quad y = B\tilde{y} \qquad \text{where} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

is composed of the new basis vectors.

Now

$$Ax = y$$

$$\Rightarrow AB\tilde{x} = B\tilde{y} \quad \Rightarrow \quad B^{-1}AB\tilde{x} = \tilde{y}$$

So $\tilde{A} = B^{-1} A B$ is the representation of the operator in the new basis,

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -5 & -2 \\ 2 & 7 & 4 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 5 \\ -2 & -2 & -3 \\ -2 & 1 & 0 \end{bmatrix}$$

Let's see if it works, for example with $x_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\tilde{x}_3 = B^{-1} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which makes sense because x_3 is the third vector in the new basis. Now,

$$\tilde{A}\tilde{x}_3 = \begin{bmatrix} 9 & 2 & 5 \\ -2 & -2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \tilde{y}$$

$$y_3 = B\tilde{y}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

which agrees with the original given value for y_3.

Prob 3 a) In the basis $\{x_1, x_2, x_3\}$, the representations of x_1, x_2, x_3 are respectively, $\tilde{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\tilde{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

There fore,

$$T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -6 \\ -1 & 1 & 3 \\ 0 & 2 & 10 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -6 \\ -1 & 1 & 3 \\ 0 & 2 & 10 \end{bmatrix}$$

b) With x_1, x_2, x_3 expressed in the standard basis, we obtain

$$T \begin{bmatrix} -2 & 1 & 8 \\ -1 & 1 & 5 \\ 2 & -1 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -6 \\ -1 & 1 & 3 \\ 0 & 2 & 10 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & -6 \\ -1 & 1 & 3 \\ 0 & 2 & 10 \end{bmatrix} \begin{bmatrix} -2 & 1 & 8 \\ -1 & 1 & 5 \\ 2 & -1 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 5 & 4 \\ -6 & 7 & -1 \\ -2 & 1 & -2 \\ 4 & 4 & 6 \end{bmatrix}$$

c) The rank$(T) = 3$, which implies T is full rank, so the degeneracy is 0.

d) The degeneracy of T is 0, so null$(T) = \{0\}$. In other words, the only x that satisfies $Tx = 0$ is $x = 0$.

e) The columns of T are linearly independent, and so form a basis for the range space of T.

Prob 4 The eigenvalues of A are 3, 1, 0 and corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.8944 \\ 0.4472 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.6667 \\ -0.6667 \\ 0.333 \end{bmatrix}$

$$\text{Trace}(A) = a_{11} + a_{22} + a_{33} = 3 + 1 + 0 = 4$$

$$\text{eigenvalues of } A: \quad \lambda_1 + \lambda_2 + \lambda_3 = 3 + 1 + 0 = 4$$

$$\det(A) = 0 \quad , \quad \lambda_1 \times \lambda_2 \times \lambda_3 = 0$$

... The eigenvalues of B are $-2, 2, 2$ and corresponding eigenvectors $\begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix}, \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$$\text{Trace}(B) = 2 \quad , \quad \lambda_1 + \lambda_2 + \lambda_3 = -2 + 2 + 2 = 2$$

$$\det(B) = -8 \quad , \quad \lambda_1 \times \lambda_2 \times \lambda_3 = -8 \; .$$

Prob 5 ... Let $\lambda = 0$, giving $\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$

Prob 6 ... Eigenvalues of A are given by $|\lambda I - A| = 0$

$$\Rightarrow \det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{bmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 1) - 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 1 = 0$$

$$\Rightarrow \lambda(\lambda - 2) = 0$$

$$\Rightarrow \lambda_1 = 0 \quad , \quad \lambda_2 = 2$$

To find eigenvectors, $(A - \lambda I) v = 0$

$$\Rightarrow \text{For } \lambda_1 = 0, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $v_1 = 1$. Then the equation is satisfied if $v_2 = -1$.

$$\text{For } \lambda_2 = 2, \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $v_1 = 1$. Then the equation is satisfied if $v_2 = 1$

Therefore,

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} , \quad V = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} , \quad V^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$V \Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$$

Eigenvalues of B are given by $|\lambda I - B| = 0$

$$\Rightarrow \det \begin{bmatrix} \lambda - 2 & -1 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda - 2)\lambda = 0$$

$$\Rightarrow \lambda_1 = 0 \ , \ \lambda_2 = 2$$

To find eigenvectors, $(B - \lambda I)\, u = 0$

For $\lambda_1 = 0$, $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $u_1 = 1$. Then the equation is satisfied if $u_2 = -2$.

For $\lambda_2 = 2$, $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $u_1 = 1$. Then $u_2 = 0$.

Therefore,

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \ V = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \ V^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$V \Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Prob 7 $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \ V = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \ V^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$$A = V \Lambda V^{-1}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$