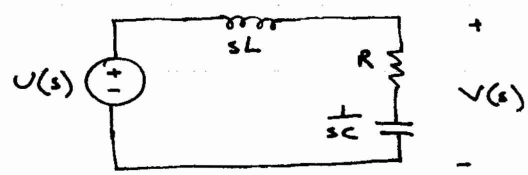
Problem Set 1

1) Transforming the circuit using Laplace transforms:



Voltage divider gives:

$$V(s) = \frac{R + \frac{1}{sC}}{R + \frac{1}{sC} + sL} \times U(s)$$
$$= \frac{RCs + 1}{RCs + 1 + LCs^2} \times U(s)$$
$$\Rightarrow (LCs^2 + RCs + 1)\, V(s) = (RCs + 1)\, U(s)$$
$$\Rightarrow \left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) V(s) = \left(\frac{R}{L}s + \frac{1}{LC}\right) U(s)$$

Taking inverse Laplace transforms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v(t) = \frac{R}{L}\frac{du}{dt} + \frac{1}{LC}u(t)$$

2) KCL gives

$$-u(t) + v_L + v_R + v_c = 0$$
$$\Rightarrow L\frac{di_L}{dt} + R i_L + v_c = u(t)$$

Also,

$$i_L = C\frac{dv_c}{dt}$$

and

$$v(t) = v_R + v_c$$
$$= R i_L + v_c$$

Writing in state space form

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

$$v(t) = \begin{bmatrix} 1 & R \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix}$$

3) The controllable canonical realization is given directly from the differential equation as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$v(t) = \begin{bmatrix} \frac{1}{LC} & \frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4) Comparing the output equations for the two state space forms, it appears that,

$$v_c = \frac{1}{LC} x_1 \quad , \quad i_L = \frac{1}{L} x_2$$

Substituting these equivalences into the state space form of part (2),

$$\frac{1}{LC} \dot{x}_1 = \frac{1}{C} * \left(\frac{1}{L} x_2\right) \qquad \Rightarrow \qquad \dot{x}_1 = x_2$$

$$\frac{1}{L} \dot{x}_2 = -\frac{1}{L} * \left(\frac{1}{LC} x_1\right) - \frac{R}{L} * \left(\frac{1}{L} x_2\right) + \frac{1}{L} u(t)$$

$$\Rightarrow \quad \dot{x}_2 = -\frac{1}{LC} x_1 - \frac{R}{L} x_2 + u(t)$$

The equations for $\dot{x}_1$ and $\dot{x}_2$ can be rewritten,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

which exactly matches the form of part (3).

(Note: In general the transformation between realizations is not so simple.)

5) The circuit is a series RLC circuit, so overdamped behaviour corresponds to

$$\alpha > \omega_0 \quad \Rightarrow \quad \frac{R}{2L} > \frac{1}{\sqrt{LC}}$$

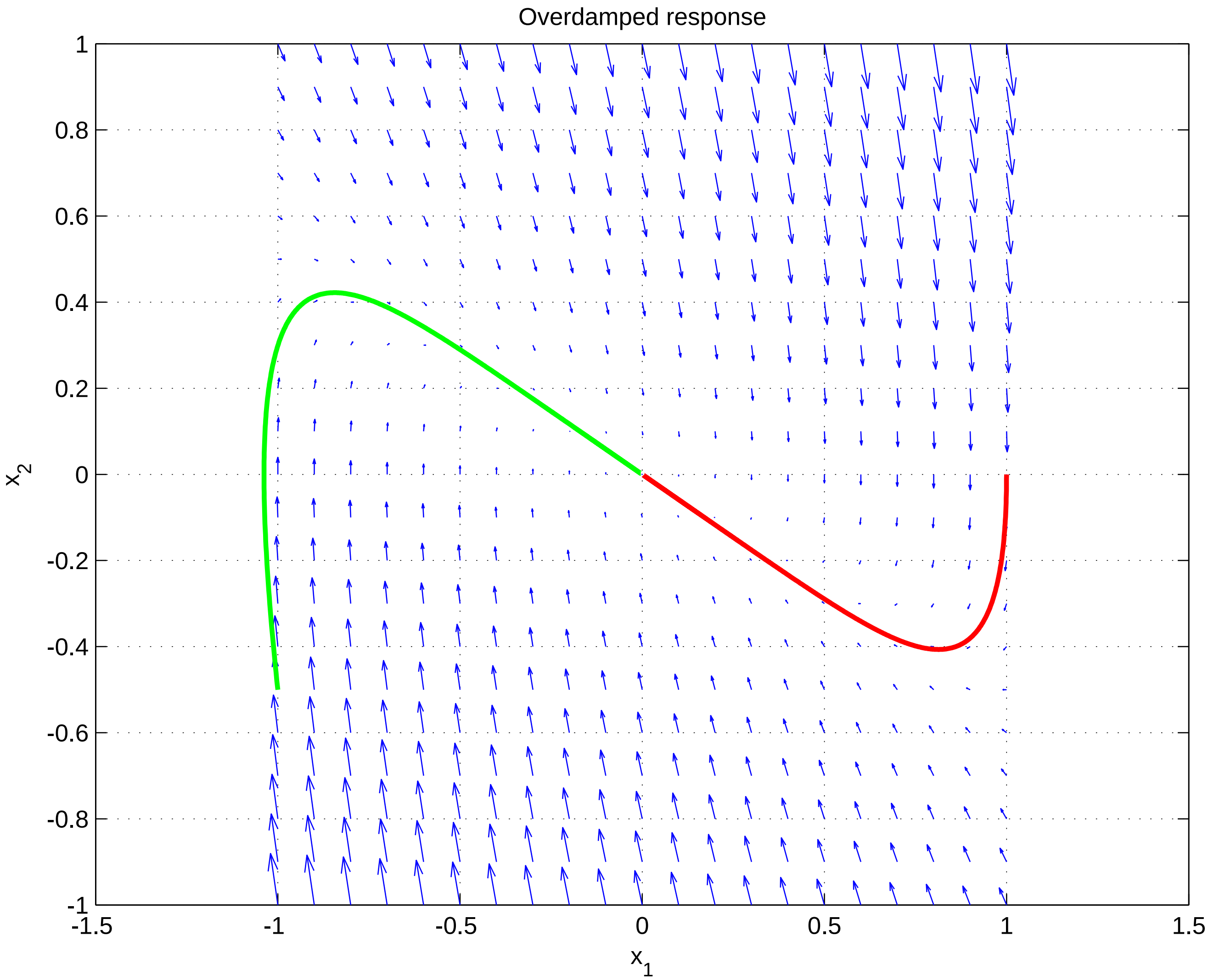
$$\Rightarrow \quad R > \frac{2L}{\sqrt{LC}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

The plot corresponds to $R = 2\ \Omega$ (overdamped)

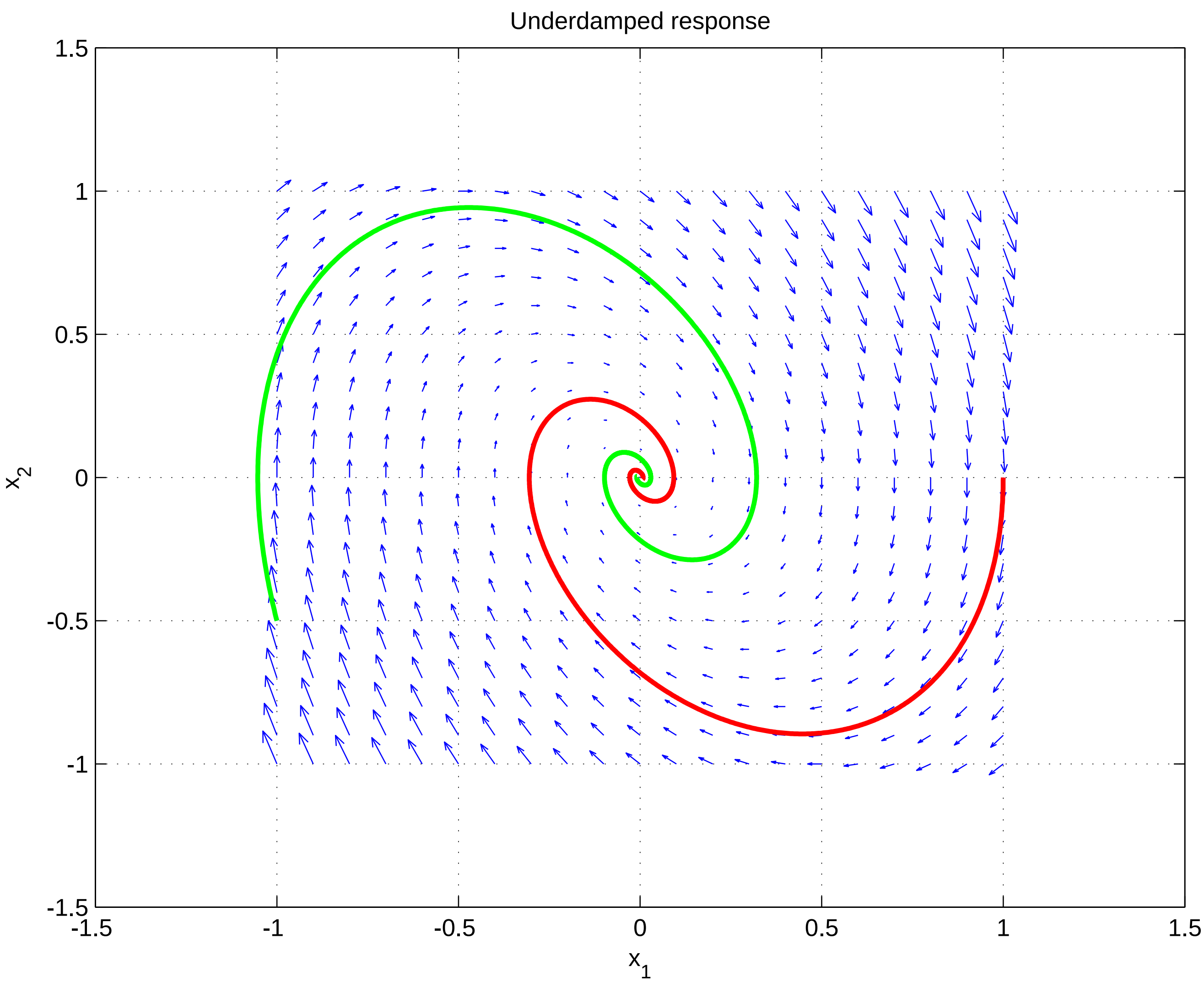
6) $R = 0.5\ \Omega$ (underdamped)

7) In the underdamped case, the trajectories swirl (oscillate) around the equilibrium point.

In the overdamped case, they approach the equilibrium point monotonically (no oscillations).



Overdamped response
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5
x_1
x_2


Underdamped response
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
x1
x2