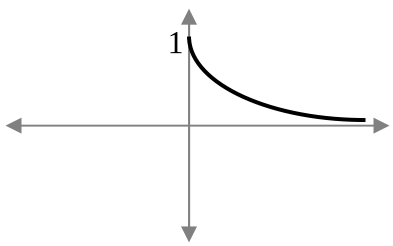
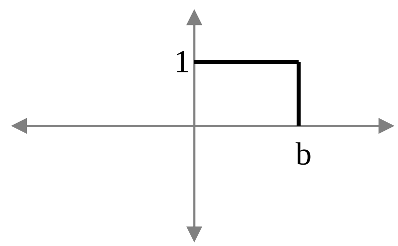
ECE 330 Summer 2006 Final Quiz Review

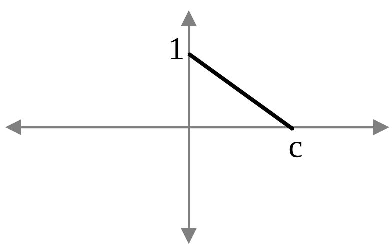
$x_1(t) = e^{-at}$



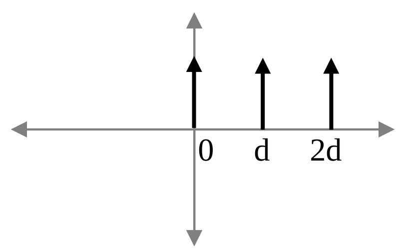
$x_2(t) = u(t) - u(t - b)$



$x_3(t) = \begin{cases} 1 - t/c, & 0 < t < c \\ 0 & \text{otherwise} \end{cases}$



$x_4(t) = \delta(t) + \delta(t - d) + \delta(t - 2d)$



Perform the following convolutions:

1. $x_1 * x_1$ a = 5 for both signals

$$x_1(t) * x_1(t) = \int_{-\infty}^{\infty} x_1(\tau)x_1(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-5\tau}u(\tau)e^{-5(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-5\tau}e^{-5(t-\tau)}d\tau$$

$$= e^{-5t}\int_0^t e^{-5\tau}e^{5\tau}d\tau = e^{-5t}[\tau]_0^t = te^{-5t}$$

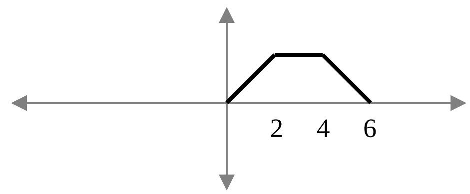
2. $x_2 * x_2$ b = 2 for one signal, 4 for the other.

For more detailed discussion on the method used to solve the following problems, see the discussion lecture notes on convolution.

$$y(0<t<2)=\int_0^t 1d\tau = t$$

$$y(2<t<4)=\int_{t-2}^t 1d\tau = [t-(t-2)] = 2$$

$$y(4<t<6)=\int_{t-2}^4 1d\tau = [4-(t-2)] = 6-t$$



3. $x_3 * x_3$ c = 2 for one signal, 3 for the other.

$$y(0<t<2)=\int_0^t\left(1-\frac{\tau}{3}\right)\left(1-\frac{t-\tau}{2}\right)d\tau=\int_0^t 1-\frac{\tau}{3}-\frac{t-\tau}{2}+\frac{\tau}{3}\frac{t-\tau}{2}d\tau=\frac{1}{6}\int_0^t 6-2\tau+3\tau-3t+t\tau-\tau^2 d\tau$$

$$=\frac{1}{6}\int_0^t 6-3t+(1+t)\tau-\tau^2 d\tau=\frac{1}{6}\int_0^t(6-3t)d\tau+\frac{1}{6}\int_0^t(1+t)\tau d\tau-\frac{1}{6}\int_0^t\tau^2 d\tau$$

$$=\left(1-\frac{t}{2}\right)[\tau]_0^t+\frac{1}{6}(1+t)\left[\frac{\tau^2}{2}\right]_0^t-\frac{1}{6}\left[\frac{\tau^3}{3}\right]_0^t=\left(t-\frac{t^2}{2}\right)+\left(\frac{t^2}{12}+\frac{t^3}{12}\right)-\frac{t^3}{18}$$

$$=\frac{1}{36}\left(36t-18t^2+3t^2+3t^3-2t^3\right)=\frac{t}{36}\left(t^2-15t+36\right)=\frac{1}{36}t(t-3)(t-12)$$

$$y(2<t<3)=\int_{t-2}^t\left(1-\frac{\tau}{3}\right)\left(1-\frac{t-\tau}{2}\right)d\tau=\left(1-\frac{t}{2}\right)[\tau]_{t-2}^t+\frac{1}{6}(1+t)\left[\frac{\tau^2}{2}\right]_{t-2}^t-\frac{1}{6}\left[\frac{\tau^3}{3}\right]_{t-2}^t$$

$$=\left(1-\frac{t}{2}\right)[t-(t-2)]+\frac{1}{6}(1+t)\left[\frac{t^2-(t-2)^2}{2}\right]-\frac{1}{6}\left[\frac{t^3-(t-2)^3}{3}\right]$$

$$=(2-t)+\frac{1}{6}(1+t)\left[\frac{t^2-\left(t^2-4t+4\right)}{2}\right]-\frac{1}{6}\left[\frac{t^3-\left(t^3-4t^2+4t-2t^2+8t-8\right)}{3}\right]$$

$$=(2-t)+\frac{1}{6}[(2t-2)+t(2t-2)]-\frac{1}{6}\left[\frac{4t^2-4t+2t^2-8t+8}{3}\right]$$

$$=(2-t)+\frac{1}{6}\left[2t^2-2\right]-\frac{1}{6}\left[2t^2-4t+\frac{8}{3}\right]=\frac{1}{18}\left(36-18t+6t^2-6-6t^2+12t-8\right)$$

$$=\frac{1}{18}(22-6t)$$

$$y(3<t<5)=\int_{t-2}^{3}\left(1-\frac{\tau}{3}\right)\left(1-\frac{t-\tau}{2}\right)d\tau=\left(1-\frac{t}{2}\right)[\tau]_{t-2}^{3}+\frac{1}{6}(1+t)\left[\frac{\tau^2}{2}\right]_{t-2}^{3}-\frac{1}{6}\left[\frac{\tau^3}{3}\right]_{t-2}^{3}$$

$$=\left(1-\frac{t}{2}\right)[3-t+2]+\frac{1}{6}(1+t)\left[\frac{9}{2}-\frac{(t-2)^2}{2}\right]-\frac{1}{6}\left[\frac{27}{3}-\frac{(t-2)^3}{3}\right]_{t-2}^{3}$$

$$=[5-t]-\left[\frac{5t}{2}-\frac{t^2}{2}\right]+\left[\frac{9-(t-2)^2}{12}\right]+\left[\frac{9t-t(t-2)^2}{12}\right]-\left[\frac{3}{2}-\frac{(t-2)^3}{18}\right]$$

$$=5-t-\frac{5t}{2}+\frac{t^2}{2}+\frac{3}{4}-\frac{t^2}{12}+\frac{t}{3}-\frac{1}{3}+\frac{3t}{4}-\frac{t^3}{12}+\frac{t^2}{3}-\frac{t}{3}-\frac{3}{2}+\frac{(t-2)(t^2-4t+4)}{18}$$

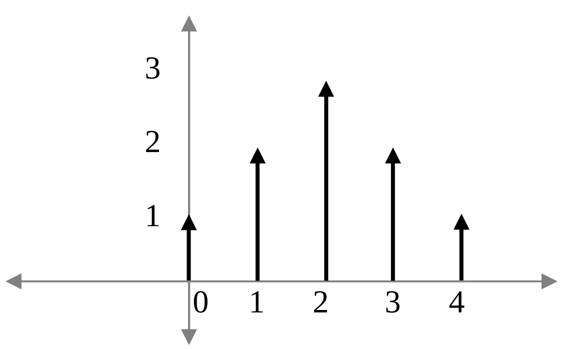
$$=5+\frac{3}{4}-\frac{1}{3}-\frac{3}{2}-\frac{8}{18}-t-\frac{5t}{2}+\frac{t}{3}+\frac{3t}{4}-\frac{t}{3}+\frac{4t}{18}+\frac{8t}{18}+\frac{t^2}{2}-\frac{t^2}{12}+\frac{t^2}{3}-\frac{4t^2}{18}-\frac{2t^2}{18}-\frac{t^3}{12}+\frac{t^3}{18}$$

$$=\frac{1}{36}\left[180+27-12-54-16-36t-90t+27t+24t+18t^2-3t^2-3t^3+2t^3\right]$$

$$=\frac{1}{36}\left[125-75t+15t^2-t^3\right]$$

Wow, that question really stunk!

4. $x_4 * x_4$ d = 1 for both signals.



5. $x_2 * x_3$ b = 3, c = 1.

$$y(0<t<1)=\int_0^t (1-\tau)d\tau=\left[\tau-\frac{\tau^2}{2}\right]_0^t=t-\frac{t^2}{2}$$

$$y(1<t<3)=\int_{t-1}^t (1-\tau)d\tau=\left[\tau-\frac{\tau^2}{2}\right]_{t-1}^t=t-\frac{t^2}{2}-\left((t-1)-\frac{(t-1)^2}{2}\right)=1-\left(\frac{2t-1}{2}\right)=\frac{3}{2}-t$$

$$y(3<t<4)=\int_{t-1}^3 (1-\tau)d\tau=\left[\tau-\frac{\tau^2}{2}\right]_{t-1}^3=3-\frac{9}{2}-\left((t-1)-\frac{(t-1)^2}{2}\right)=-\frac{3}{2}-\left(t-1-\frac{t^2-2t+1}{2}\right)$$

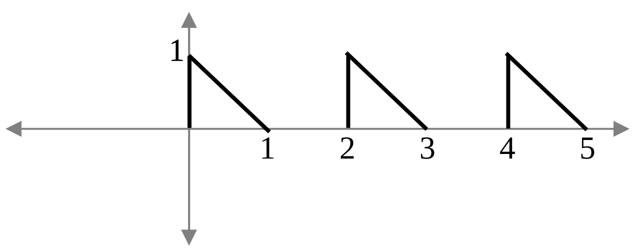
$$=-\frac{1}{2}-t+\frac{t^2}{2}-t+\frac{1}{2}=-2t+\frac{t^2}{2}$$

6. $x_1 * x_2$ a = 1, b = 3.

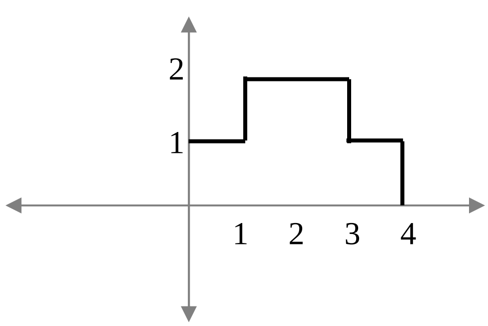
$$y(0<t<3)=\int_0^t e^{-\tau}d\tau=\left[-e^{-\tau}\right]_0^t=1-e^{-t}$$

$$y(3<t<\infty)=\int_{t-3}^t e^{-\tau}d\tau=\left[-e^{-\tau}\right]_{t-3}^t=-e^{-t}+e^{-(t-3)}=e^{-t}\left(e^3-1\right)$$

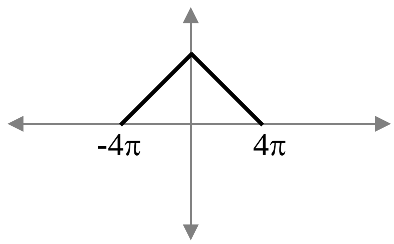
7. $x_3 * x_4$ c = 1, d = 2.



8. $x_2 * x_4$ b = 2, d = 1.



9. A continuous time signal x(t) has the Fourier transform $X(\omega)$ below:

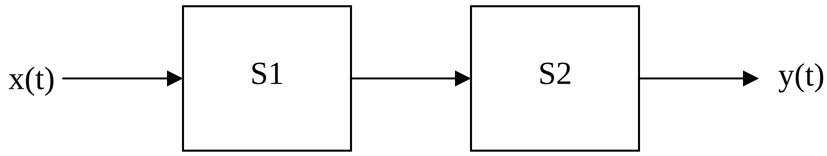


(measured in radians, which is 2π/seconds).

How often (how many times per second) must we sample the time signal in order to be able to reconstruct it perfectly from its Fourier transform? In other words, how often must we sample x(t) so that its discrete Fourier transform $X(\Omega)$ does not experience any aliasing? In other words, what is the Nyquist frequency of x(t)?

The Nyquist frequency is twice as large as the highest frequency needed to represent the signal. In our signal, $|\omega|_{max} = 4\pi$. It is more useful to deal in Hertz, and the relationship is $\omega = 2\pi f$. Therefore 4π radians = 2 Hertz, or 2 per second. We must sample at twice that rate in order to capture all the relevant info about our signal, which means we have to sample **4 times per second**.

10. A signal $x(t)$ is passed through two systems S1 and S2. The impulse response for S1 is $h_1(t)$ and the impulse response for S2 is $h_2(t)$.



If $h_1(t) = x_2(t)$ from page 1, with $b = 1$,
$h_2(t) = x_4(t)$ from page 1, with $d = 1$,
and $x(t) = x_3(t)$ from page 1, with $c = 1$,

show why $y(t)$ is the same as the answer for problem #5.

We can find the output of S1 by calculating $x(t) * h_1(t)$. We then convolve that by $h_2(t)$ to get $y(t)$. In other words, $[\ x(t) * h_1(t)\ ] * h_2(t) = y(t)$.

If we take that into the frequency domain, the convolution problems become multiplication problems: $[\ X(\omega)\ H_1(\omega)\ ]\ H_2(\omega) = Y(\omega)$. The associative property states that $[\ X(\omega)\ H_1(\omega)\ ]\ H_2(\omega) = X(\omega)\ [\ H_1(\omega)\ H_2(\omega)\ ] = Y(\omega)$. Taking that back into the time domain, $x(t) * [\ h_1(t) * h_2(t)\ ] = y(t)$.

$h_1(t) * h_2(t)$ is a square pulse of length 3, which is the same as x_2 in problem 5. $x(t)$ is the same as x_3 from problem 5. Thus we see that it is the same convolution problem.